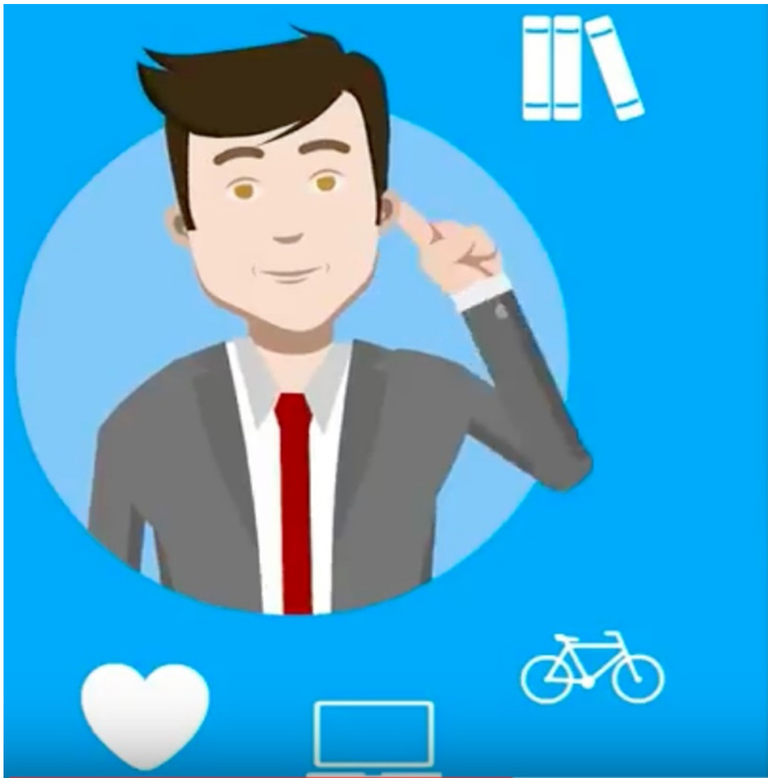

STUDY CHALLENGE

STUDY TIPS

SAT Math

with

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Class 16 : Geometry 1

The SAT Math: Additional Topics

What other special topics are included on the SAT Math test?

About 10% (6 out of 58 points) of the SAT Math questions are "Additional Topics" questions. These include topics like

- analyzing triangles using the Pythagorean Theorem
- graphing circles and other figures in the *xy*-plane
- analyzing areas, circumferences, chords, and sectors of circles
- measuring angles and arcs in radians
- working with area and volume and their formulas
- using the theorems of congruence and similarity
- working with basic trigonometric relationships including cofunction identities
- calculating with imaginary and complex numbers

Why are these topics important?

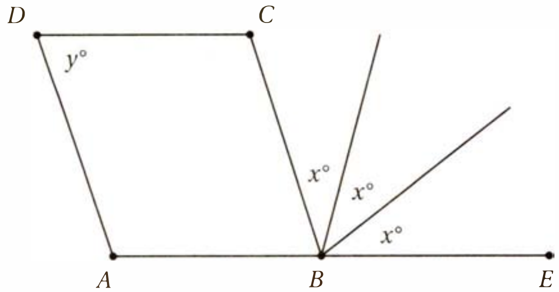
These topics from geometry, trigonometry, and advanced analysis are crucial to work in engineering, physics, architecture, and even design. Although they are not essential to every college major, they do provide tools for understanding and analyzing advanced concepts across the curriculum.

Sound intimidating? It's not.

Some of you have already spent some time in math class studying these topics. If not, the three skills described in these 12 lessons will give you the knowledge and practice you need to master them.

Skill 1: Understanding Geometric Relationships

Lesson 1: Intersecting and parallel lines



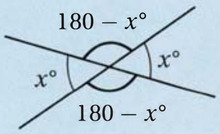
In the figure above, $ABCD$ is a parallelogram, and point B lies on $\overline{AE}$. If $x = 40$, what is the value of y?

A) 40 B) 50 C) 60 D) 70

(*Medium*) Since $ABCD$ is a parallelogram, we can take advantage of the Parallel Lines Theorem.

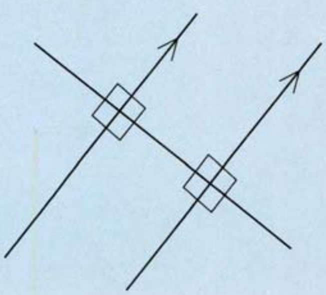
The Intersecting Lines Theorem

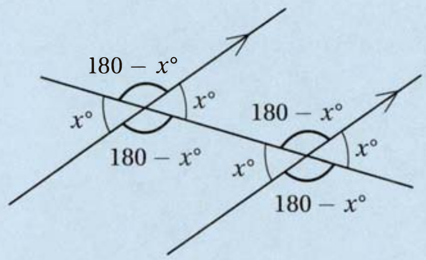
When two lines cross, four angles are formed. The vertical angles are congruent and adjacent angles are supplementary (that is, they have a sum of 180°).



The Parallel Lines Theorem

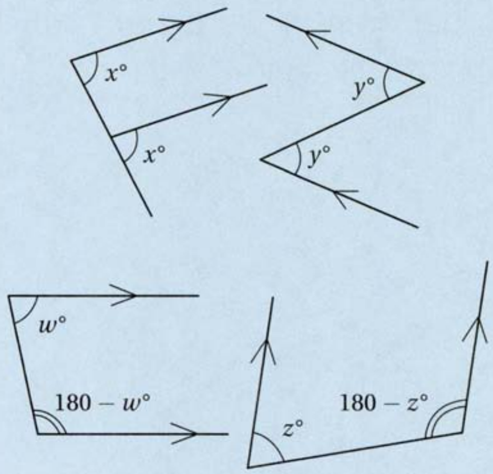
When two parallel lines are crossed by a third line, eight angles are formed. If the third line is perpendicular to one of the parallel lines, then it's perpendicular to the other and all eight angles are right angles. Otherwise, all four acute angles are congruent, all four obtuse angles are congruent, and any acute angle is supplementary to any obtuse angle.



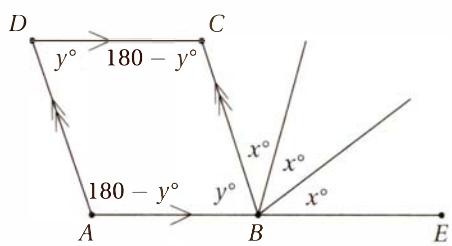


Helpful Tip

When dealing with parallel lines, especially in complicated figures, we can simplify things by considering angles in pairs. The important pairs form one of four letters: F, Z, C, or U.

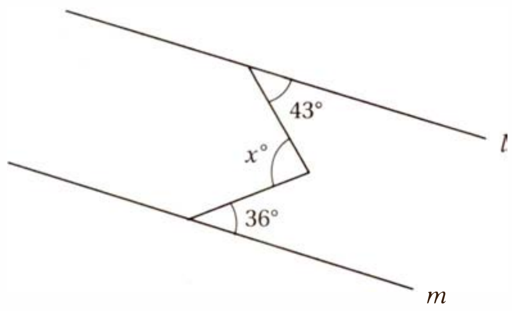


First, let's mark up the diagram with what we know from the Parallel Lines Theorem.



Since the pairs of opposite sides are parallel, the consecutive angles in the parallelogram must be supplementary (that is, have a sum of 180°). Notice that these pairs of consecutive angles form "U"s or "C"s as mentioned in the previous Helpful Tip. This implies that **opposite angles are congruent** in a parallelogram.

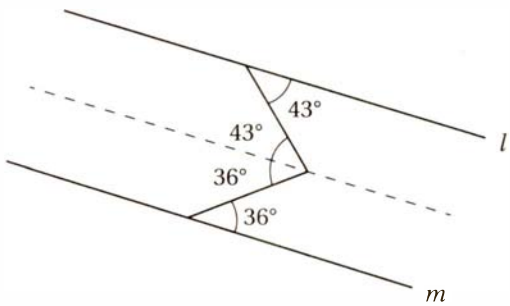
Since $\overline{ABE}$ is a straight (180°) angle:
$y + x + x + x = 180$
Substitute $x = 40$ and simplify: $y + 120 = 180$
Subtract 120: $y = 60$
Therefore, the correct answer is (C).



In the figure above, lines l and m are parallel. What is the value of x?

A) 43 B) 79 C) 86 D) 101

(*Hard*) Although our diagram includes parallel lines, it doesn't seem to show any of the parallel line "letter pairs" that we discussed above, because no line directly connects the parallel lines. We can fix this problem by drawing an extra line that's parallel to l and m through the vertex of the angle.

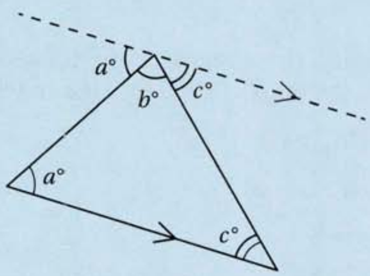


Now we have two "Z" pairs of angles (otherwise known as "alternate interior" pairs) that show that the middle angle is actually the sum of two smaller angles of 36° and 43°, and therefore, $x = 36 + 43 = 79$, and the correct answer is (B).

Lesson 2: Triangles

Angle Sum Theorem

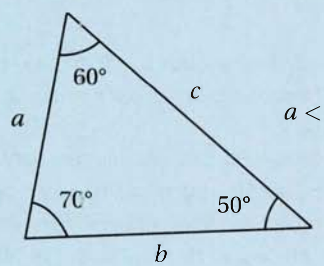
The sum of the measures of the angles in any triangle is 180°.



We can prove this with the "draw an extra line" trick. If we take any triangle, pick any of its vertices, and draw a line through that vertex that is parallel to the opposite side, we get a picture like the one above. Since the line we've drawn is a 180° angle, and since the "Z" angle pairs must be congruent, we've proven that $a + b + c = 180$.

Side-Angle Theorem

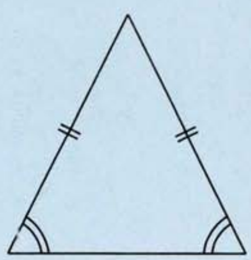
The largest angle in a triangle is always across from the largest side, and the smallest angle is always across from the smallest side.



$a < b < c$

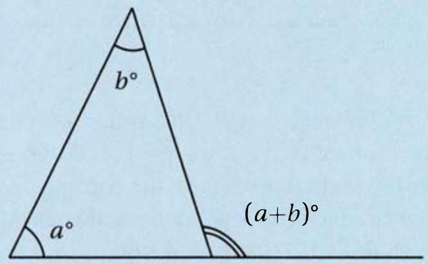
Isosceles Triangle Theorem

If two sides in a triangle are congruent, the two angles across from those sides are also congruent. Conversely, if two angles in a triangle are congruent, the two sides across from them are also congruent.



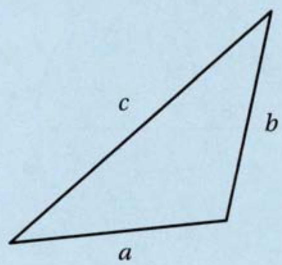
Exterior Angle Theorem

If the side of a triangle is extended beyond a vertex, it makes an exterior angle with the adjacent side. The measure of this exterior angle is equal to the sum of the two remote interior angles.



The Triangle Inequality

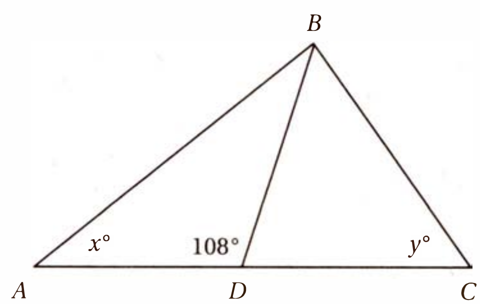
The sum of any two sides of a triangle must always be greater than the third side.



$a + b > c$

$b + c > a$

$c + a > b$



In the figure above, if $AD = DB = DC$, what is the value of $x + y$?

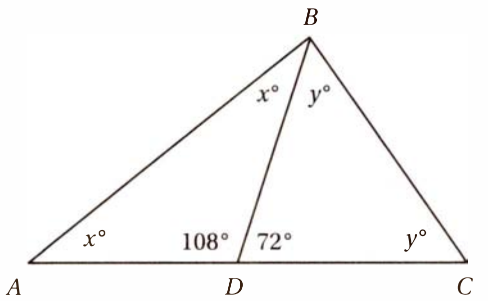
A) 72

B) 90

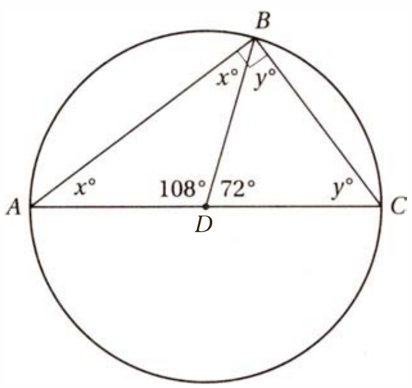
C) 96

D) 108

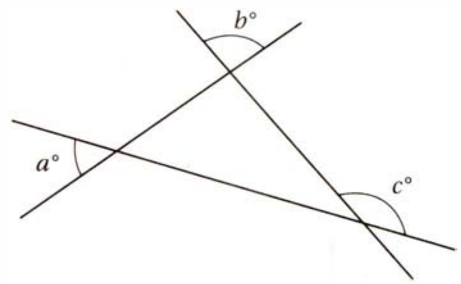
(*Medium*) Since angle ADB and angle BDC are supplementary and $AD = DB = DC$, we can take advantage of the Isosceles Triangle Theorem to mark up the diagram.



Now let's look at triangle ABC. Since its interior angles must have a sum of 180°, $x + x + y + y = 180$, and therefore, $2x + 2y = 180$ and $x + y = 90$. So the correct answer is (B). Notice that this fact is independent of the measures of the other two (108° and 72°) angles. As long as $AD = DB = DC$, this relationship will hold. We can see these angle relationships if we notice that these three segments could all be radii of a circle centered at D.



You may remember from studying geometry that any "inscribed" angle (an angle inside a circle with a vertex on the circle) intercepts an arc on the circle that is twice its measure. Since angle ABC is an inscribed angle that intercepts a 180° arc, it must have a measure of 90° and therefore, $x + y = 90$.



The figure above shows three intersecting lines. What is the value of c in terms of a and b?

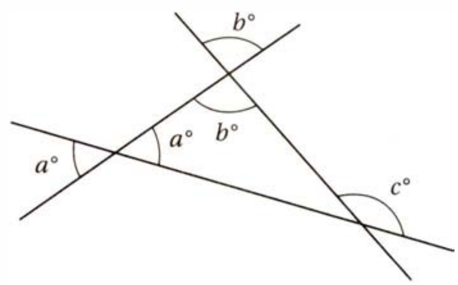
A) $180 - a - b$

B) $180 - a + b$

C) $90 + b - a$

D) $a + b$

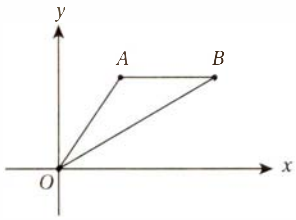
(*Easy*) First, we should notice that two of the angles are "vertical" to two interior angles of the triangle, and the other is an exterior angle.



Since the $c°$ angle is an exterior angle to the triangle, the Exterior Angle Theorem tells us that $c = a + b$, so the correct answer is (D).

Alternately, we could just choose reasonable values for a and b, like $a = 50$ and $b = 90$, and then analyze the diagram in terms of these values. This would imply that the interior angles of the triangle are 50°, 90°, and 40°, and $c°$ would then be the measure of the supplement of 40°, which is 140°. If we then plug these values for a and b into all of the choices, the only one that yields 140 is D.

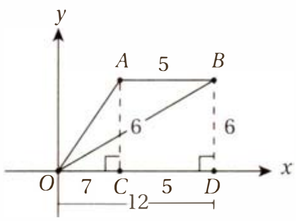
Lesson 3: The *xy*-plane



Note: Figure not drawn to scale.

In the *xy*-plane above, points *A* and *B* lie on the graph of the line $y = 6$. If *OB* has a slope of $\frac{1}{2}$ and $AB = 5$, what is the slope of $\overline{OA}$?

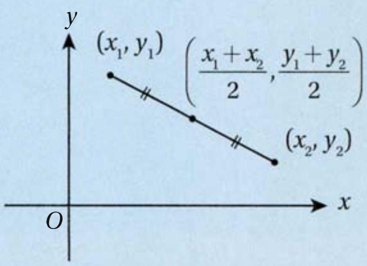
(*Medium-hard*) To analyze this diagram, we must recall the definition of slope from Chapter 7, Lesson 5.



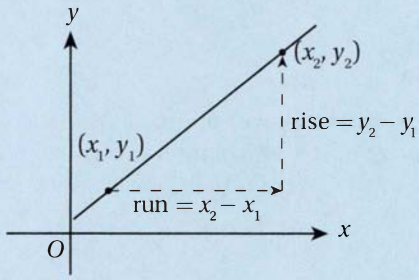
First, let's drop two perpendicular segments from *A* and *B* to points *C* and *D*, respectively, on the *x*-axis. Since *A* and *B* lie on the line $y = 6$, they are both 6 units from the *x*-axis, and so $AC = BD = 6$. Then, since the slope of *OB* is ½, $BD/OD = ½$, and therefore, $OD = 12$. Since $AB = 5$, $CD = 5$ also, and therefore, $OC = 12 - 5 = 7$. (Don't worry that $\overline{OC}$ looks shorter than $\overline{CD}$ in the diagram. Remember, the figure is not drawn to scale!) This gives us everything we need to find the slope of $\overline{OA}$, which connects (0, 0) to (7, 6). By the slope formula from Chapter 7, Lesson 5, slope $= (6 - 0)/(7 - 0) = 6/7 = 0.857$.

Working in the Coordinate Plane

Finding Segment Midpoints. To find the coordinates of a **midpoint**, just average the coordinates of the **endpoints**.

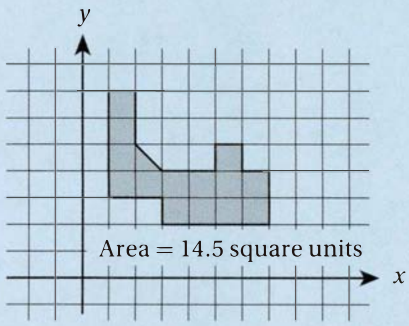


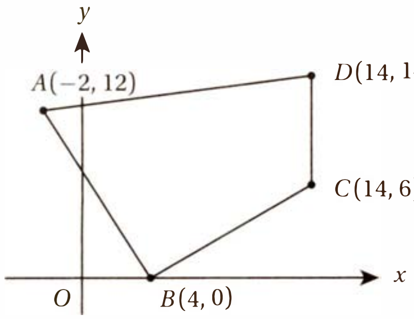
Finding Slopes. To find the **slope** of a line in the *xy*-plane from any two points on the line, use the **slope formula**.



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Finding Areas. Remember that the **area** of a figure is just the number of **unit squares** that fit inside it. You don't always need to use a special formula to find the area of a figure. Even for very complicated shapes, you can sometimes find the area just by counting squares.



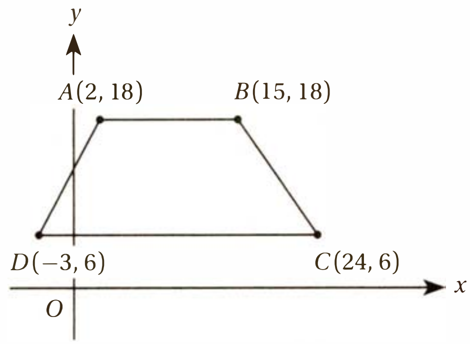


In the figure above, point M (not shown) is the midpoint of $\overline{AB}$ and point N (not shown) is the midpoint of $\overline{CD}$. What is the slope of $\overline{MN}$?

(*Medium*) To find the midpoint of a segment, we just need to take the average of the endpoints. Point M, the midpoint of $\overline{AB}$, therefore has coordinates $\left(\frac{-2+4}{2}, \frac{12+0}{2}\right) = (1,6)$, and point N, the midpoint of $\overline{CD}$, has coordinates $\left(\frac{14+14}{2}, \frac{6+14}{2}\right) = (14,10)$.

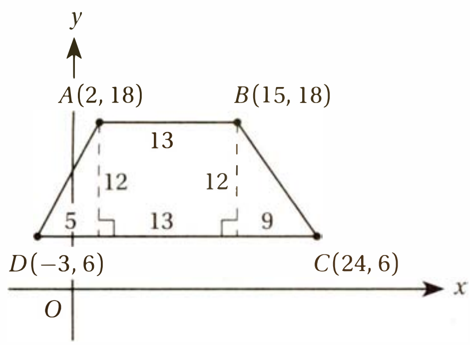
By the Slope Formula, then, the slope of $\overline{MN}$ is $\frac{10-6}{14-1} = \frac{4}{13} = 0.307$ or 0.308.

Lesson 4: The Pythagorean Theorem and the Distance Formula



What is the perimeter of quadrilateral *ABCD* in the figure above?

(*Medium*) The perimeter of a figure is the distance around its edges. It's easy to find the lengths of $\overline{AB}$ and $\overline{DC}$ because they are horizontal. The length of a horizontal segment is just the difference between the *x*-coordinates of its endpoints. The length of $\overline{AB}$ is $15 - 2 = 13$, and the length of $\overline{DC}$ is $24 - (-3) = 27$. To find the lengths of $\overline{AD}$ and $\overline{BC}$, we can drop two vertical lines from points *A* and *B* to the bottom edge. This shows that $\overline{AD}$ and $\overline{BC}$ are hypotenuses of two right triangles as shown in the figure below.

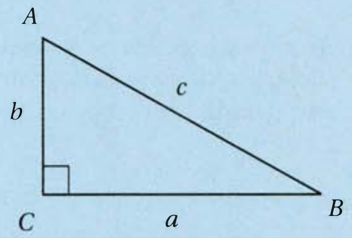


(Take a minute to confirm the lengths of all the segments for yourself.) With this information, we can find *AD* and *BC* by the Pythagorean Theorem.

The Pythagorean Theorem

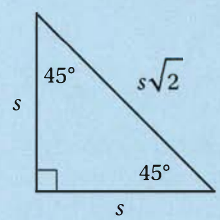
If *a*, *b*, and *c* represent the sides of a right triangle in which *c* is the longest side (the hypotenuse),

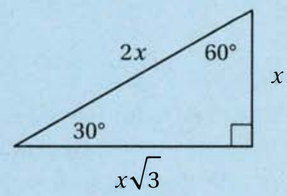
$$a^2 + b^2 = c^2$$

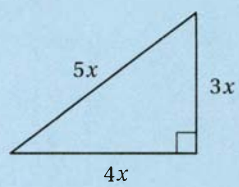


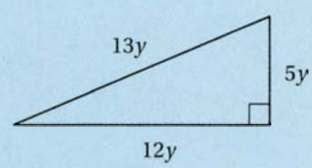
Special Right Triangles

The SAT Math test expects you to be familiar with four families of special right triangles: **45°-45°-90° triangles**, **30°-60°-90° triangles**, **3-4-5 triangles**, and **5-12-13 triangles**. Take some time to familiarize yourself with these particular side-side relationships and side-angle relationships so that you can use these relationships when you recognize these triangles in SAT Math questions.









$(3x)^2 + (4x)^2 = (5x)^2$

$9x^2 + 16x^2 = 25x^2$

$(5y)^2 + (12y)^2 = (13y)^2$

$25y^2 + 144y^2 = 169y^2$

So, according to our diagram: $AD^2 = 5^2 + 12^2 = 169$

$BC^2 = 9^2 + 12^2 = 225$

Notice that triangle on the left is a 5-12-13 special right triangle, and the triangle on the right is a 3-4-5 special right triangle. Noticing these relationships provides a shortcut to using the Pythagorean Theorem.

Take the square root: $AD = 13$

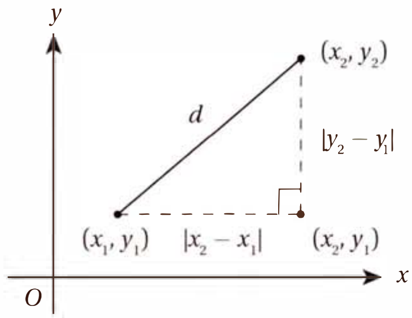
$BC = 15$

Therefore, the perimeter of $ABCD$ is $13 + 15 + 27 + 13 = 68$.

The Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We can generalize the technique we used in the previous problem to find the distance between *any* two points in the *xy*-plane. Just think of this distance as the length of the hypotenuse of a right triangle, as in the figure below. In other words, the Pythagorean Theorem and the Distance Formula are one and the same.



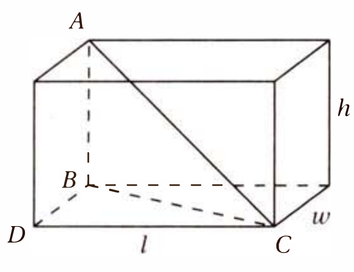
By the Pythagorean Theorem: $|x_2 - x_1|^2 + |y_2 - y_1|^2 = d^2$

Take the square root: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The 3-D Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

If we need to find the distance between two points in three-dimensional *xyz*-space, we just need to use a modified version of the distance formula that includes the extra *z*-dimension. You can see where this formula comes from if you imagine trying to find the length of the longest diagonal through a rectangular box.



The length of this diagonal, AC, is also the hypotenuse of right triangle ABC, and so its length is given by the Pythagorean Theorem.

Pythagorean Theorem for ABC: $AC = \sqrt{(AB)^2 + (BC)^2}$

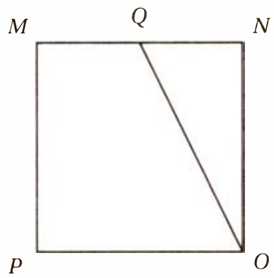
Pythagorean Theorem for BDC: $(BC)^2 = (BD)^2 + (DC)^2$

Substitute: $AC = \sqrt{(AB)^2 + (BD)^2 + (DC)^2}$

Since $AB = h$, $BD = w$, and $DC = l$ $\quad AC = \sqrt{l^2 + w^2 + h^2}$

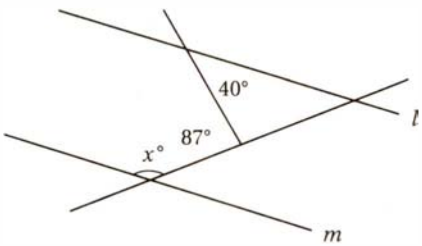
Exercise Set 1: Geometry (No Calculator)

1



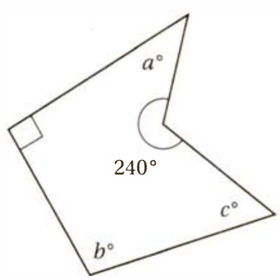
In the figure above, *MNOP* is a square and *Q* is the midpoint of $\overline{MN}$. If $QO = \frac{\sqrt{20}}{3}$, what is the area of square *MNOP*?

2



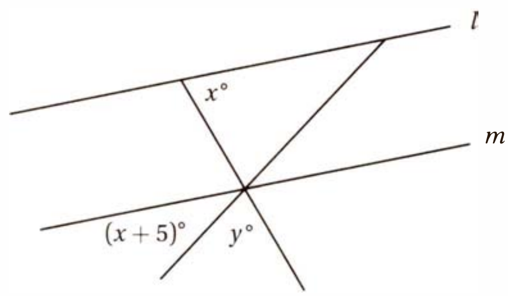
Lines *l* and *m* are parallel in the figure above. What is the value of *x*?

3



In the figure above, what is the value of $a + b + c$?

4



Lines *l* and *m* are parallel in the figure above. Which of the following expresses the value of *y* in terms of *x*?

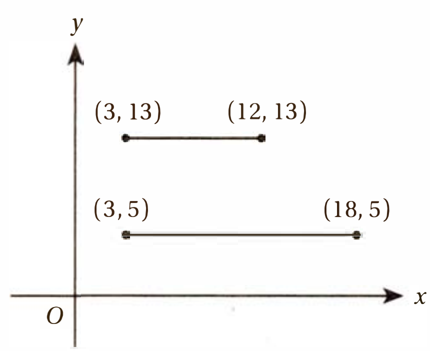
A) $95 - 2x$

B) $165 - 2x$

C) $175 - 2x$

D) $185 - 2x$

5



In the figure above, what is the distance between the midpoints (not shown) of the two line segments?

A) $\sqrt{68}$ B) $\sqrt{73}$ C) $\sqrt{76}$

D) $\sqrt{78}$

6

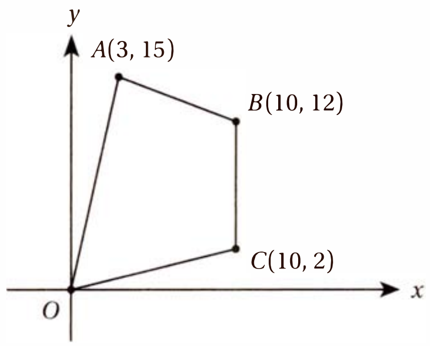
What is the perimeter of an equilateral triangle inscribed in a circle with circumference 24π?

A) $36\sqrt{2}$ B) $30\sqrt{3}$

C) $36\sqrt{3}$ D) $24\sqrt{6}$

Exercise Set 1: Geometry (Calculator)

Questions 7–9 are based on the figure below.



Note: Figure not drawn to scale.

7

In the figure above, what is the perimeter of quadrilateral $ABCO$, to the nearest integer?

8

In the figure above, what is the area, in square units, of $ABCO$?

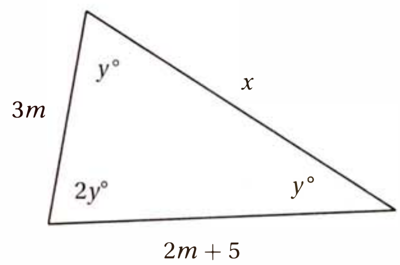
9

In the figure above, point K (not shown) is the midpoint of $\overline{OA}$, and point M (not shown) is the midpoint of $\overline{AB}$. What is the slope of $\overline{KM}$?

10

In the xy-plane, point H has coordinates (2, 1) and point J has coordinates (11, 13). If $\overline{HK}$ is parallel to the x-axis and $\overline{JK}$ is parallel to the y-axis, what it the perimeter of triangle HJK?

11



Note: Figure not drawn to scale.

In the figure above, what is the value of x?

A) $5\sqrt{2}$ B) $5\sqrt{3}$ C) $15\sqrt{2}$ D) $15\sqrt{3}$

Questions 12–15 are based on the situation described below.

In the xy-plane, $ABCD$ is a square. Point A has coordinates $(-1, 2)$ and point B has coordinates (3, 5).

12

Which of the following could be the coordinates of C?

A) (0, 9) B) (6, 0) C) $(2, -2)$ D) $(-4, 6)$

13

What is the area of square $ABCD$?

A) 25 B) 28 C) 30 D) 32

14

What is the slope of $\overline{BC}$?

A) $-\frac{4}{3}$ B) $-\frac{3}{4}$ C) $\frac{3}{4}$ D) $\frac{4}{3}$

15

What is the distance between C and the midpoint of $\overline{AB}$?

A) $\frac{\sqrt{50}}{4}$ B) $\frac{\sqrt{125}}{4}$ C) $\frac{\sqrt{50}}{2}$

D) $\frac{\sqrt{125}}{2}$

EXERCISE SET 1: GEOMETRY ANSWER KEY

No Calculator

1. **16/9 or 1.77 or 1.78** If we define x as the length of $\overline{QN}$, then the length of one side of the square is $2x$, and so the area of square $MNOP$ is $(2x)(2x) = 4x^2$. To find this value, we can apply the Pythagorean Theorem to right triangle QNO:

$$x^2 + (2x)^2 = \left(\frac{\sqrt{20}}{3}\right)^2$$

Simplify: $$5x^2 = \frac{20}{9}$$

Divide by 5: $$x^2 = \frac{20}{45} = \frac{4}{9}$$

Multiply by 4: $$4x^2 = \frac{16}{9} = 1.77 \text{ or } 1.78$$

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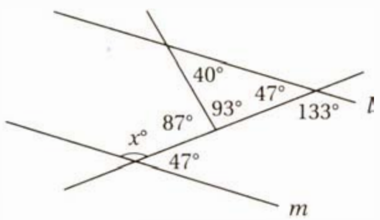
2. **133** The key is to notice simple relationships between angles until we get around to x.



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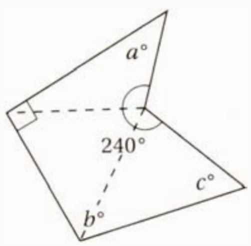
3. **210** Draw three lines as shown:



Since the polygon divides into 3 triangles, the sum of its internal angles is $(3)(180°) = 540°$. Therefore $a + b + c + 240 + 90 = 540$, and so $a + b + c = 210$.

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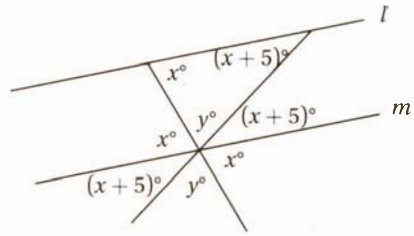
4. **C** Using the Crossed Lines Theorem and the Parallel Lines Theorem, we can mark up the diagram like this:



This shows that $x + y + x + 5 = 180$, and so $y = 175 - 2x$.

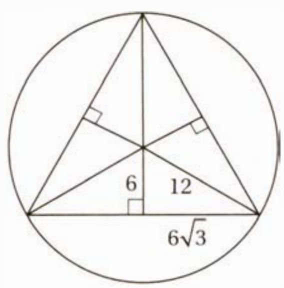
---

5. **B** The midpoint of the top segment is $\left(\frac{3+12}{2}, \frac{13+13}{2}\right) = \left(\frac{15}{2}, 13\right)$, and the midpoint of the bottom segment is $\left(\frac{3+18}{2}, \frac{5+5}{2}\right) = \left(\frac{21}{2}, 5\right)$; therefore, the distance between them is

$$\sqrt{\left(\frac{21}{2} - \frac{15}{2}\right)^2 + (13-5)^2} = \sqrt{3^2 + 8^2} = \sqrt{73}$$

---

6. **C** To solve this problem we must draw a diagram and find the relationship between the radius of the circle and the sides of the triangle. By the Isosceles Triangle Theorem, if all three sides of a triangle are congruent, then all three angles must be congruent. Since these angles also must have a sum of 180°, they must each be 60°. If we draw the bisectors of each of these angles, we divide the triangle into six smaller triangles. These smaller triangles are congruent 30°-60°-90° triangles, as shown here:



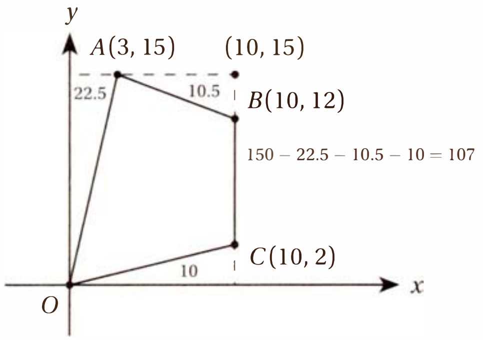
Since the circumference of the circle $(2\pi r)$ is 24π, its radius is 12. Since each of the hypotenuses of our right triangles is also a radius of the circle, we can find all of the sides of these triangles using the 30°-60°-90° relationships. Each side of the equilateral triangle is therefore $2\left(6\sqrt{3}\right) = 12\sqrt{3}$, and its perimeter is therefore $2\left(12\sqrt{3}\right) = 36\sqrt{3}$.

Calculator

7. **43** Using the distance formula, we can calculate the lengths of each segment. $OA = \sqrt{234} \approx 15.30$, $AB = \sqrt{58} \approx 7.61$, $BC = 10$, and $OC = \sqrt{104} \approx 10.20$. Therefore, the perimeter is approximately $15.30 + 7.61 + 10 + 10.20 = 43.11$, which rounds to 43.

---

8. **107** Since we do not have a formula that directly calculates the area of such an odd-shaped quadrilateral, we must analyze its area in terms of simpler shapes. The simplest way to do this is by drawing a box around it. This turns the area of interest into a rectangle minus three right triangles, all of which have areas that can be easily calculated.



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9. **6/5 or 1.2** The midpoint of $\overline{OA}$ is (1.5, 7.5) and the midpoint of $\overline{AB}$ is (6.5, 13.5); therefore, the slope of the segment between them is 6/5.

---

10. **36** If point K is on the same horizontal line as (2, 1), it must have a y-coordinate of 1, and if it is on the same vertical line as (11, 13), it must have an x-coordinate of 11. Therefore, K is the point (11, 1), and so $HK = 9$, $JK = 12$, and $HJ = \sqrt{9^2 + 12^2} = \sqrt{225} = 15$. Notice that it is a 3-4-5 triangle!

---

11. **C** Since the sum of the interior angles of any triangle is 180°, $y + y + 2y = 4y = 180$, and therefore $y = 45$. Therefore, this is a 45°-45°-90° right triangle. Since two angles are equal, the two opposite sides must also be equal, so $3m = 2m + 5$ and so $m = 5$ and the two legs each have measure 15. Using the Pythagorean Theorem or the 45°-45°-90° shortcut, we can see that $x = 15\sqrt{2}$.

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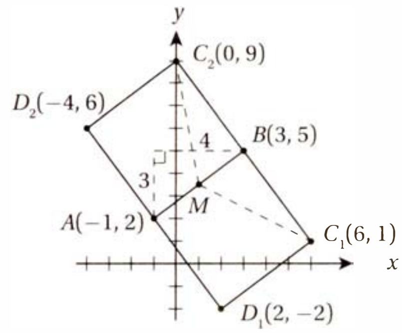
12. **A** The key to questions 12 through 15 is a good diagram in the xy-plane that represents the given information:



If $ABCD$ is a square, then the points A, B, C, and D must appear *in that order* around the square. Notice that to get from point A to point B, we must move 4 units to the right and 3 units up. This means that, in order to get to point C along a perpendicular of the same length, we must go either *3 units right and 4 units down*, or *3 units left and 4 units up*. This puts us either at (6, 1) or (0, 9).

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13. **A** The diagram shows that AB is the length of the hypotenuse of a right triangle with legs 3 and 4. You should recognize this as the special 3-4-5 right triangle. If $AB = 5$, then the area of the square is $5^2 = 25$.

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14. **A** Notice that the slope of $\overline{BC}$ is the same regardless of which option we choose for C. In either case, the slope formula tells us that the slope is $-4/3$.

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15. **D** The midpoint of $\overline{AB}$ (point M above) is (1, 3.5). We can use the distance formula to find the distance between this point and either of the possible locations of C. (Notice that the distance is the same either way.) Alternately, we might notice that MC is the hypotenuse of a right triangle with legs 5 and 2.5. Either way, we get a value of $\frac{\sqrt{125}}{2}$.